Introduction to Dynamical Systems

Solutions Problem Set 2

Exercise 1. Let T be a continuous map on a metric space X without isolated points. Show that if x has a dense orbit under T, then so does $T^n(x)$ for every $n \geq 1$.

Solution. We can use the Birkhoff Transitivity Theorem. Let $\varepsilon > 0$ and $y \in X$, and for $n \geq 1$ and $x \in X$ with a dense orbit, consider the ball $U = B_{\varepsilon/2}(T^n(x))$. Taking $V = B_{\varepsilon/2}(y)$, the theorem establishes that we may find $k \in \mathbb{N}$ for which

$$T^k(U) \cap V \neq \emptyset.$$

In particular, this implies that $d(T^n(x), y) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. By the arbitrariness of ε and $y \in X$, we conclude. $\square$

Exercise 2. Suppose that X is a metric space with at least one isolated point. Then if $T\colon X \longrightarrow X$ is continuous and such that for any non-empty open sets U, V there is $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$, then X is a finite set and $X = \mathcal{O}^+(x)$.

Solution. Let $x \in X$ be an isolated point, and $\varepsilon > 0$ small enough such that $B_\varepsilon(x) = \{x\}$. Notice that by the hypotheses, the orbit $\mathcal{O}^+(x)$ is dense in X, as for any open set $V \subset X$,

$$\mathcal{O}^+(x) \cap V = \bigcup_{n \geq 0} T^n(B_\varepsilon(x)) \cap V \neq \emptyset.$$

Hence there is $n \geq 0$ such that $T^n(x) \cap T^{-1}(B_\varepsilon(x)) \neq \emptyset$, meaning that $T^{n+1}(x) = x$, and therefore the orbit $\mathcal{O}^+(x)$ is periodic. Finally, since $\mathcal{O}^+(x)$ is a dense, finite (and therefore closed) set, we conclude that $X = \overline{\mathcal{O}^+(x)} = \mathcal{O}^+(x)$. $\square$

Exercise 3. By considering the set of 2^n-th roots of unity, $n \geq 1$, show that the second part of the Birkhoff transitivity theorem is violated if we drop the extra assumptions on (X, d).

Solution. We let

$$X = \left\{ e^{2\pi i \frac{q}{2^n}} : q \in \left\{0, \ldots, 2^{n-1}\right\},\, n \in \mathbb{N} \right\} \subset S^1,$$

equipped with the usual distance, and consider the map $T(x) = x^2$. Notice that $T^k(U)$ evolves into $S^1 \cap X$ if U is an open set as $k \to \infty$. In particular, letting U, V be two nonempty, open sets, we may choose $x = e^{2\pi i q/2^n} \in V$ and $e^{2\pi i r/2^k} \in U$ for certain $q, r, n, k \in \mathbb{N}$. Then, by letting ℓ be large enough and since U is open, we have

$$y = e^{2\pi i \left(\frac{r}{2^k} + \frac{q}{2^{n+k+\ell}}\right)} \in U.$$

We immediately find $T^{k+\ell}(y) = x$. Despite this, no element has a dense orbit, since every orbit ends up at $T^j(x) = 1$ for $j \in \mathbb{N}$ large enough. $\square$